INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

Note: this is a Calculus III final exam. While some problems may be solved using techniques from Calculus I or II, you may not receive full credit if you do so, even if the final result is correct.

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1. (25 points) Determine the x-coordinate of the center of mass, $\bar{x}$, of a thin plate with density (mass per unit area) given by $\delta(x, y) = 1 + x$, bounded by the curves $y = x^2$ and $y = x(2 - x)$ in the first quadrant. Recall that

$$\bar{x} = \iint_R x\,\delta\,dA \,\Big/ \iint_R \delta\,dA\,.$$

2. (25 points) After making a fortune tutoring desperate Calculus students, Karen bought a nice piece of property in Montana. Since she loves star gazing, she built a large glass dome so she can look at the stars on cold nights. The surface of the dome is given by $z = 9 - (x^2 + y^2)$ for $z \geq 0$. One evening at sunset Karen is in her dome and a large bird flies over head. It unloads right on the top of her dome. She watches the deposit run down the side of the dome and create a long white streak. Having well calibrated eyes, and a good head for Calculus III, she quickly determines the path of the deposit as it slides down the surface of the dome. Specifically, the path of the payload is given by $\mathbf{r}(t) = t\,\mathbf{i} + 0\,\mathbf{j} + f(t)\,\mathbf{k}$, for $t \geq 0$.

   (a) Determine the function $f(t)$ if the payload was at the top of the dome when $t = 0$. If you cannot determine $f(t)$, you may purchase it at a cost of 5 points. This offer expires at 6:00 pm.

   (b) At what time does the deposit hit the ground?

   (c) How far has the deposit travelled along the surface of the dome?

   (d) Next to the dome is a fence, the surface of which is described by $x = 10$. If one were to project a line, L, from the origin (at center of the floor of the dome), through the drop's location when $t = 2$, determine the coordinates of the point where the line intersects the fence.

   (e) Determine an expression for the angle between the line L and the direction of motion of the drop when $t = 2$.

   (f) It costs more to wash the top of the dome than the lower portions. Specifically $C = 3z$ where C is the cost in dollars per foot. You may assume all the variables x, y, and z are in feet. Set up, but do not evaluate, the calculations the cost of cleaning the surface of the dome.

3. (25 points) Consider a spherical loaf of bread with radius R. Suppose you cut a slice from the loaf between the planes $y = a$ and $y = b$, where $0 < a < b < R$.

   (a) Set up, but do not evaluate, the integral that will give you the volume of the slice of bread.

   (b) Suppose that $b = a + t$ where t is the thickness of the slice. Show that the amount of crust on the slice does not depend on the value of a.

4. (25 points) Consider the integral

$$I = \iint_{R_{xy}} 8xy \; dx \, dy \,,$$

where R_{xy} is the region in the xy-plane bounded by the curves $x = 0$, $y = x$, $y = 1 - x$, and $y = 2 - x$.

(a) The substitution $u = x + y$ and $v = x - y$ will simplify the evaluation of I. Find x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!

(b) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make two clear sketches, one of the original region of integration R_{xy} in the xy-plane, and one of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc. on each sketch.

(c) Rewrite the integral for I over the region R_{uv} in the uv-plane in terms of u and v.

(d) Evaluate I in terms of u and v.

5. (25 points) Use the transformation $u = 3x + 2y$ and $v = x + 4y$ to evaluate the integral

$$I = \iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

over the region R in the first quadrant bounded by the lines $y = -\frac{3}{2}x + 1$, $y = -\frac{3}{2}x + 3$, $y = -\frac{1}{4}x$, and $y = -\frac{1}{4}x + 1$.

(a) Solve for x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!

(b) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make a clear sketch of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc. on your sketch.

(c) Rewrite the integral for I over the region R_{uv} in the uv-plane in terms of u and v.

(d) Evaluate I in terms of u and v.